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$1/s$-expansion for generalized dimensions in a hierarchical s-state Potts model

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Abstract. We consider the s-state Potts model on the diamond hierarchical lattice for large s. We show that the generalized dimensions D_q of the density of zeros supported by the associated Julia set are given by $D_q = 1 - (q-1)|s|^{-2/3}/4\log 2 + O(s^{-1})$. The information dimension D_1 equals 1 to all orders.

1. Introduction

The diamond hierarchical lattice is constructed recursively by replacing each bond at one step by a set of four bonds in a diamond shape as shown in figure 1. The beauty of this lattice is that renormalization (in the sense of statistical mechanics) may be carried out exactly. Using this fact, Derrida *et al* [1] show that the Yang–Lee zeros of the s-state Potts model on this lattice are dense in the Julia set of the rational map

$$z \mapsto \left(\frac{z^2+s-1}{2z+s-2}\right)^2 . \tag{1.1}$$

When s is large the Julia set of this map is a Jordan circle, and, as observed by Hu and Lin [2], as $s \to \infty$, it becomes larger and more circular. Figure 2 shows the Julia sets for $s = 10$, 20 and 30. We shall carry out an s-dependent scaling of this Julia set to make this apparent, and see that (in our new coordinates) in the limit $s \to \infty$ the map becomes the simpler map

$$z \mapsto z^4 . \tag{1.2}$$

We shall consider perturbations about this limit and derive an expression for the generalized dimensions D_q of the density of zeros supported by the associated Julia set. The calculations are similar to those recently carried out for the degree-d complex map

$$z \mapsto z^d + c \tag{1.3}$$

for small c. In [3] it is shown that

$$D_q = 1 - \frac{(q-1)|c|^2}{4\log d} + \delta_{d,2}\frac{(q-1)(q-3)(c^2\bar{c}+\bar{c}^2 c)}{16\log d} + \cdots$$

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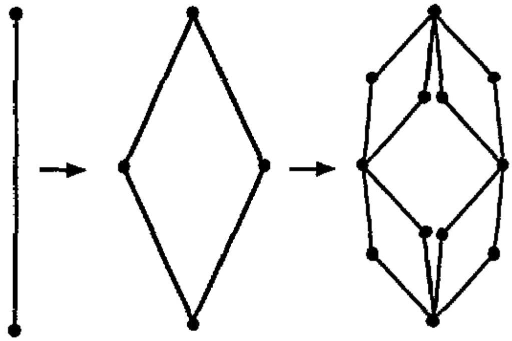

Figure 1. The first few steps in the construction of the diamond hierarchical lattice.

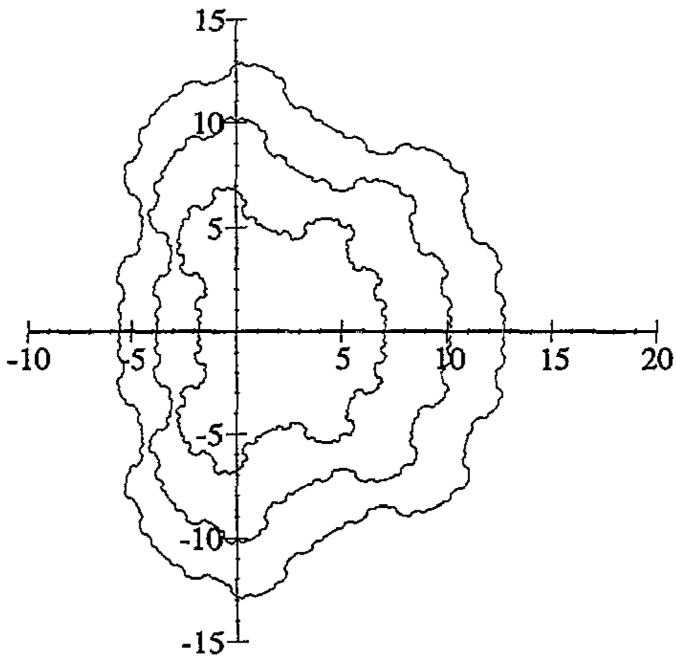

Figure 2. The Julia sets of the rational map (1.1) with (from the inside) $s = 10$, 20 and 30.

in which $\delta_{i,j}$ is the Kronecker delta function

$$\delta_{i,j} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise} . \end{cases}$$

This generalized an earlier result of Ruelle [4] for the Hausdorff dimension D_0:

$$D_0 = 1 + \frac{|c|^2}{4 \log d} + \cdots .$$

Ruelle's result was extended by Widom *et al* [5] who show

$$D_0 = 1 + \frac{|c|^2}{4 \log d} + \delta_{d,2} \frac{3(c^2 \bar{c} + \bar{c}^2 c)}{16 \log d} + \cdots .$$

Independently, and by different means, Collet *et al* [6] have also derived the result in [3], and indeed calculated the next order term.

For the Julia set associated with the diamond hierarchical lattice we show here that

$$D_q = 1 - \frac{(q-1)|s|^{-2/3}}{4\log 2} + O(s^{-1}).$$

In particular, the Hausdorff dimension D_0 has expansion $1 + |s|^{-2/3}/4\log 2 + O(s^{-1})$. The information dimension D_1 is equal to 1 to all orders in $1/s$.

Unlike the result for the map (1.3) the term of order s^{-1} cannot be argued away on the grounds of symmetry, despite the degree of the map (1.1) being four.

Associated with the generalized dimensions by means of a Legendre transform is the $f(\alpha)$ singularity spectrum [7]. To the level of our series approximation we show that this has the parabolic form

$$f(\alpha) = 1 + (\alpha - 1) - \frac{(\alpha-1)^2 \log 2}{|s|^{-2/3}}$$

about $s = \infty$.

In principle, terms of successively higher order may be calculated. However, the algebraic manipulations rapidly get more complicated.

2. The rescaled map

For $s > \frac{32}{27}$ the map (1.1) has an unstable fixed point at $z = z_*$ where

$$z_* = 1 + s_1 + s_2$$

with

$$s_1 = \left(r + \sqrt{\Delta}\right)^{1/3} \qquad \text{and} \qquad s_2 = \left(r - \sqrt{\Delta}\right)^{1/3}$$

where

$$r = \tfrac{1}{2}s^2 \qquad \text{and} \qquad \Delta = s^3\left(\tfrac{1}{4}s - \tfrac{8}{27}\right).$$

To see this notice that $z = 1$ is always a fixed point of (1.1) enabling us to factorize the fixed-point equation and leave a cubic to solve. When $s > \frac{32}{27}$ this cubic has one real root which is an unstable fixed point.

We now change coordinates to move this point to 1 by setting $y = z/z_*$. This gives the map

$$y \mapsto g(y) = \frac{1}{z_*}\left(\frac{z_*^2 y^2 + s - 1}{2z_* y + s - 2}\right)^2.$$

The function g may be expanded as a series in powers of $s^{-1/3}$ and, setting $p = s^{-1/3}$, we find

$$g(y) = y^4 + pC_1(y) + p^2 C_2(y) + O(p^3) \tag{2.1}$$

where

$$C_1(y) = 2y^2 + 2y^4 - 4y^5$$
$$= -2y^2(y-1)\left(2y^2+y+1\right)$$

and

$$C_2(y) = 1 + \tfrac{4}{3}y^2 - 8y^3 + \tfrac{13}{3}y^4 - \tfrac{32}{3}y^5 + 12y^6$$
$$= \tfrac{1}{3}(y-1)\left(36y^5 + 4y^4 + 17y^3 - 7y^2 - 3y - 3\right).$$

Generally C_n is a polynomial of degree $n+4$ with a zero at unity. It may be calculated by tedious but straightforward algebra.

3. Conjugation

When $p = 0$ the Julia set is the unit circle which we may parametrize in the form $y(t) = \exp(2\pi i t)$. The map (1.2) is then simply $t \mapsto 4t \bmod 1$. Equivalently, we may say that y satisfies the conjugacy equation

$$g(y(t)) = y(4t). \tag{3.1}$$

For small p the map (2.1) is conjugate to the case $p = 0$ so that (3.1) still holds. y is analytic in the parameter p and so we may formally write

$$y(t) = \mathrm{e}^{2\pi i t}\left(1 + pu_1(t) + p^2u_2(t) + \mathrm{O}(p^3)\right). \tag{3.2}$$

Following Widom *et al* [5], the expansion (3.2) may be substituted into (3.1), where we set

$$\delta = \mathrm{e}^{2\pi i t} \tag{3.3}$$

to yield

$$u_1(4t) - 4u_1(t) = \delta^{-4}C_1(\delta) \tag{3.4}$$

and

$$u_2(4t) - 4u_2(t) = 6u_1(t)^2 + \delta^{-3}C_1'(\delta)u_1(t) + \delta^{-4}C_2(\delta) \tag{3.5}$$

(where the 'prime' denotes a derivative). Now the basic linear equation

$$\psi(4t) - 4\psi(t) = \mathrm{e}^{-2\pi i k t} \tag{3.6}$$

has a solution $\phi(kt)$, where

$$\phi(t) = -\tfrac{1}{4}\sum_{\ell=0}^{\infty} 4^{-\ell}\,\mathrm{e}^{-2\pi i 4^{\ell} t}. \tag{3.7}$$

Hence equation (3.4) has the solution

$$u_1(t) = 2\phi(2t) + 2\phi(0) - 4\phi(-t). \tag{3.8}$$

(We could, of course, write $\phi(0) = -\frac{1}{3}$.)

For u_2 we find (solving for the three terms on the right-hand side of (3.5) separately)

$$u_2(t) = u_2^{(1)}(t) + u_2^{(2)}(t) + u_2^{(3)}(t) \tag{3.9}$$

where

$$\begin{aligned} u_2^{(1)}(t) = \tfrac{3}{2} \sum_{\ell,m=0}^{\infty} 4^{-(\ell+m)} \big(&\phi((2.4^\ell + 2.4^m)t) + \phi(0) + 4\phi((-4^\ell - 4^m)t) \\ &+ 2\phi(2.4^\ell t) - 4\phi((2.4^\ell - 4^m)t) - 4\phi(-4^\ell t)\big) \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_2^{(2)}(t) = -2 \sum_{\ell=0}^{\infty} 4^{-\ell} \big(&\phi((2.4^\ell + 2)t) + 2\phi(2.4^\ell t) - 5\phi((2.4^\ell - 1)t) \\ &+ \phi(2t) + 2\phi(0) - 5\phi(-t) \\ &- 2\phi((-4^\ell + 2)t) - 4\phi(-4^\ell t) + 10\phi((-4^\ell - 1)t)\big) \end{aligned} \tag{3.11}$$

and

$$u_2^{(3)}(t) = \phi(4t) + \tfrac{4}{3}\phi(2t) - 8\phi(t) + \tfrac{13}{3}\phi(0) - \tfrac{32}{3}\phi(-t) + 12\phi(-2t)\,. \tag{3.12}$$

4. Multifractal analysis

The multifractal analysis (see for instance [7]) of the 'strange' dynamics on a set such as a Julia set consists of partitioning of the set into balls such that ball j has length ℓ_j and an associated probability measure p_j. One forms the partition function

$$\Gamma(q, \tau) = \sum_j p_j^q / \ell_j^\tau \tag{4.1}$$

and, fixing q and finding the supremum (infimum) of Γ for $q > 1$ (respectively $q < 1$) over all partitions, one finds that, in the limit $\max \ell_j \to 0$, Γ is of order unity for just one value of τ. This defines the function $\tau(q)$, and the generalized dimensions D_q are then defined by

$$\tau(q) = (q-1) D_q\,. \tag{4.2}$$

In particular, setting $q = 0$ we recover the Hausdorff dimension. Setting $q = 1$ (2) we have the information (respectively correlation) dimension.

A Legendre transform of this function from variables q and τ to α and f gives the so-called $f(\alpha)$ or multifractal spectrum:

$$\alpha = \frac{d\tau}{dq} \qquad f(\alpha) = q\alpha - \tau\,. \tag{4.3}$$

Here, in the partition function (4.1), the lengths we use are the distances between successive members of $\mathcal{F}_n$, the set of period points of g of period n:

$$\mathcal{F}_n = \operatorname{Fix} g^n = \{y : g^n(y) = y\}\,.$$

In the case $p = 0$ these are simply the points $y_j = \exp(2\pi \mathrm{i} t_j)$ where $t_j = j/(4^n - 1)$, $j = 0, 1, \ldots, 4^n - 2$. When p is non-zero there is no closed-form expression for the members of $\mathcal{F}_n$ and we rely on the series expansion (3.2) for small p obtained via the conjugacy (3.1). We have

$$\ell_j = |y_{j+1} - y_j| = |y(t_j + \omega) - y(t_j)| \qquad j = 0, 1, \ldots, 4^n - 2$$

where $\omega = 1/(4^n - 1)$.

We assign equal weight to each length in the partition function, and because of this uniformity we calculate the function $q(\tau)$ rather than its inverse $\tau(q)$. Let $N = 4^n - 1$. Setting $\Gamma = 1$ and $p_j = 1/N$ in (4.1), we see that

$$q_n(\tau) = \frac{1}{\log N} \log \sum_{j=0}^{N-1} \ell_j^{-\tau} \tag{4.4}$$

where $q_n(\tau)$ is the nth approximation to the limiting function $q(\tau)$.

5. The calculation

In equation (4.4) we need to evaluate the sum $\sum_{j=0}^{N-1} \ell_j^{-\tau}$. From equation (3.2) we may write

$$\begin{aligned} y(t+\omega) - y(t) &= \mathrm{e}^{2\pi \mathrm{i} t} \left(\mathrm{e}^{2\pi \mathrm{i}\omega} - 1 + \left(\mathrm{e}^{2\pi \mathrm{i}\omega} u_1(t+\omega) - u_1(t)\right) p \right. \\ &\qquad \left. + \left(\mathrm{e}^{2\pi \mathrm{i}\omega} u_2(t+\omega) - u_2(t)\right) p^2 + \mathrm{O}(p^3)\right) \\ &= \mathrm{e}^{2\pi \mathrm{i} t} \left(A + B(t) p + C(t) p^2 + \mathrm{O}(p^3)\right) \end{aligned} \tag{5.1}$$

say, thereby defining the constant A and the functions B and C.

We have

$$\begin{aligned} \ell_j^{-\tau} &= |y(t_j + \omega) - y(t_j)|^{-\tau} \\ &= ((y(t_j + \omega) - y(t_j))(\bar{y}(t_j + \omega) - \bar{y}(t_j)))^{-\tau/2} \\ &= \left(A\bar{A} + (A\bar{B}\bar{p} + \bar{A}Bp) + (A\bar{C}\bar{p}^2 + B\bar{B}p\bar{p} + \bar{A}Cp^2) + \mathrm{O}(p^3)\right)^{-\tau/2} \\ &= (A\bar{A})^{-\tau/2} \left(1 - \frac{\tau}{2} \frac{(A\bar{B}\bar{p} + \bar{A}Bp)}{A\bar{A}} - \frac{\tau}{2} \frac{\left(A\bar{C}\bar{p}^2 + B\bar{B}p\bar{p} + \bar{A}Cp^2\right)}{A\bar{A}} \right. \\ &\qquad \left. + \frac{1}{2}\frac{\tau}{2}\left(\frac{\tau}{2} + 1\right) \left(\frac{A\bar{B}\bar{p} + \bar{A}Bp}{A\bar{A}}\right)^2 + \mathrm{O}(p^3)\right). \end{aligned}$$

Following Widom *et al* [5] again it is convenient to define the average

$$\langle G(t)\rangle_n = \frac{1}{N} \sum_{j=0}^{N-1} G(t_j)\,. \tag{5.2}$$

A fundamental property of this average is

$$\langle e^{2\pi i m t}\rangle_n = \begin{cases} 1 & m \equiv 0 \pmod N \\ 0 & m \not\equiv 0 \pmod N\,. \end{cases} \tag{5.3}$$

Note also that this average is linear and that

$$\langle G(t+\omega)\rangle_n = \langle G(t)\rangle_n \qquad \text{and} \qquad \langle \bar{G}(t)\rangle_n = \overline{\langle G(t)\rangle}_n\,. \tag{5.4}$$

Thus to second order we have

$$\begin{aligned}\sum_{j=0}^{N-1} \ell_j^{-\tau} &= N\langle \ell_j^{-\tau}\rangle_n \\ &= N(A\bar{A})^{-\tau/2}\Bigg(\langle 1\rangle_n - \frac{\tau}{2}\left(\bar{p}\frac{\langle \bar{B}\rangle_n}{\bar{A}} + p\frac{\langle B\rangle_n}{A}\right) \\ &\quad + p^2\left(-\frac{\tau}{2}\frac{\langle C\rangle_n}{A} + \frac{1}{2}\frac{\tau}{2}\left(\frac{\tau}{2}+1\right)\frac{\langle B^2\rangle_n}{A^2}\right) \\ &\quad + \bar{p}^2\left(-\frac{\tau}{2}\frac{\langle \bar{C}\rangle_n}{\bar{A}} + \frac{1}{2}\frac{\tau}{2}\left(\frac{\tau}{2}+1\right)\frac{\langle \bar{B}^2\rangle_n}{\bar{A}^2}\right) \\ &\quad + p\bar{p}\left(\left(\frac{\tau}{2}\right)^2\frac{\langle B\bar{B}\rangle_n}{A\bar{A}}\right)\Bigg).\end{aligned} \tag{5.5}$$

Now from (5.1) $A = e^{2\pi i\omega} - 1$, and, recalling that $\omega = 1/(4^n - 1) = 1/N$, so

$$\begin{aligned} A\bar{A} &= (e^{2\pi i\omega} - 1)(e^{-2\pi i\omega} - 1) = 2(1 - \cos 2\pi\omega) \\ &= \frac{4\pi^2}{N^2} + O(1/N^4)\,. \end{aligned}$$

Hence

$$\log(A\bar{A}) = -2\log N + O(1) \qquad \text{as} \quad N \to \infty\,. \tag{5.6}$$

In the appendix we show

$$\frac{\langle B\rangle_n}{A} = -\frac{2}{3} \qquad \frac{\langle C\rangle_n}{A} = -\frac{5}{9} \qquad \frac{\langle B^2\rangle_n}{A^2} = \frac{4}{9} \tag{5.7}$$

and

$$\frac{\langle B\bar{B}\rangle_n}{A\bar{A}} = \frac{\log N}{\log 2} + O(1) \qquad \text{as} \quad N \to \infty\,. \tag{5.8}$$

Hence, using (5.6)–(5.8) and the immediate fact that $\langle 1\rangle_n = 1$, our expansion to second order (5.5) becomes

$$\begin{aligned} q_n(\tau) &= \frac{1}{\log N}\log\sum_{j=0}^{N-1}\ell_j^{-\tau} \\ &= \frac{1}{\log N}\left(\log\left(N(A\bar{A})^{-\tau/2}\right) + \log\left(1 + p\bar{p}\frac{\tau^2\log N}{4\log 2} + O(1)\right)\right) \\ &= \frac{1}{\log N}\left(\log N - \frac{\tau}{2}\log(A\bar{A})\right) + \frac{1}{\log N}\left(p\bar{p}\frac{\tau^2\log N}{4\log 2} + O(1)\right) \\ &= \tau + 1 + p\bar{p}\frac{\tau^2}{4\log 2} + O(1/\log N) \end{aligned}$$

and thus

$$q(\tau) = \lim_{n\to\infty} q_n(\tau) = \tau + 1 + p\bar{p}\frac{\tau^2}{4\log 2} + O(p^3).$$

Inverting this series gives

$$\tau(q) = q - 1 - p\bar{p}\frac{(q-1)^2}{4\log 2} + O(p^3)$$

and hence by (4.2)

$$D_q = \frac{1}{q-1}\tau(q) = 1 - p\bar{p}\frac{(q-1)}{4\log 2} + O(p^3).$$

For the multifractal spectrum we have (by equation (4.3))

$$\alpha = \frac{d\tau}{dq} = 1 - p\bar{p}\frac{(q-1)}{2\log 2} + O(p^3)$$

and hence to second order

$$f(\alpha) = q\alpha - \tau = 1 + (\alpha - 1) - (\alpha - 1)^2\frac{\log 2}{p\bar{p}}.$$

Recalling that $p = s^{-1/3}$ we get the results as stated in the introduction.

Appendix A. $\langle B\rangle_n$, $\langle B^2\rangle_n$ and $\langle C\rangle_n$

We shall use the notation

$$e(k) = e^{-2\pi ikt}$$

so that $e(p)e(q) = e(p+q)$, $\delta^p = e(-p)$ (see equation (3.3)), and (referring to (3.6) and (3.7)) the linear equation

$$\psi(4t) - 4\psi(t) = e(k)$$

has solution $\phi(kt)$, where

$$\phi(t) = -\tfrac{1}{4}\sum_{\ell=0}^{\infty} 4^{-\ell} e(4^\ell).$$

Equation (5.3) may be written

$$\langle e(m)\rangle_n = \begin{cases} 1 & m \equiv 0 \pmod{4^n - 1} \\ 0 & m \not\equiv 0 \pmod{4^n - 1}. \end{cases} \qquad \text{(A1)}$$

From equation (5.1)

$$\langle B\rangle_n = \langle \gamma u_1(t+\omega) - u_1(t)\rangle_n = \gamma\langle u_1(t+\omega)\rangle_n - \langle u_1(t)\rangle_n.$$

Now from (3.8)

$$\begin{aligned}\langle u_1(t)\rangle_n &= 2\langle\phi(2t)\rangle_n + 2\langle\phi(0)\rangle_n - 4\langle\phi(-t)\rangle_n \\ &= -\tfrac{1}{2}\sum_{\ell=0}^{\infty} 4^{-\ell}\left(\langle e(2.4^\ell)\rangle_n + \langle e(0)\rangle_n - 2\langle e(-4^\ell)\rangle_n\right).\end{aligned}$$

Hence by (A1), since $2.4^\ell \not\equiv 0 \pmod{4^n-1}$ and $-4^\ell \not\equiv 0 \pmod{4^n-1}$, and clearly $\langle e(0)\rangle_n = 1$, we have

$$\langle u_1(t)\rangle_n = -\tfrac{1}{2}\sum_{\ell=0}^{\infty} 4^{-\ell} = 2\phi(0) = -\tfrac{2}{3}.$$

(To deduce that $2.4^\ell \not\equiv 0 \pmod{4^n-1}$ and $-4^\ell \not\equiv 0 \pmod{4^n-1}$ it suffices to note that $2.4^\ell \equiv 2 \pmod 3$ and $-4^\ell \equiv 2 \pmod 3$.) By equation (5.4) and the fact that $A = \gamma - 1$ we immediately see that

$$\frac{\langle B\rangle_n}{A} = -\frac{2}{3}.$$

The evaluation of $\langle B^2\rangle_n$ is similar. From equation (5.1), $B^2 = \gamma^2 u_1(t+\omega)^2 + u_1(t)^2 - 2\gamma u_1(t+\omega)u_1(t)$. Note that

$$u_1(t+\omega) = -\tfrac{1}{2}\sum_{\ell=0}^{\infty} 4^{-\ell}\left(\mathrm{e}^{-2\pi \mathrm{i}4^\ell 2\omega} e(2.4^\ell) + e(0) - 2\,\mathrm{e}^{2\pi \mathrm{i}4^\ell\omega} e(-4^\ell)\right).$$

When we form $\langle u_1(t+\omega)u_1(t)\rangle_n$ as above we see that there is only one non-zero term (that given by $e(0)$) and we easily deduce that $\langle u_1(t+\omega)u_1(t)\rangle_n = \frac{4}{9}$. (To see that all other terms vanish it suffices to consider the arguments modulo 3.) The other two terms in $\langle B^2\rangle_n$ are the same and we immediately deduce that

$$\frac{\langle B^2\rangle_n}{A^2} = \frac{4}{9}.$$

For $\langle C\rangle_n$ we have from (5.1)

$$\langle C\rangle_n = \gamma\langle u_2(t+\omega)\rangle_n - \langle u_2(t)\rangle_n$$

and from (3.9)

$$\langle u_2(t)\rangle_n = \langle u_2^{(1)}(t)\rangle_n + \langle u_2^{(2)}(t)\rangle_n + \langle u_2^{(3)}(t)\rangle_n.$$

We shall consider these three terms separately.

From equation (3.12)

$$\langle u_2^{(3)}(t)\rangle_n = \langle\phi(4t)\rangle_n + \tfrac{4}{3}\langle\phi(2t)\rangle_n - 8\langle\phi(t)\rangle_n + \tfrac{13}{3}\langle\phi(0)\rangle_n - \tfrac{32}{3}\langle\phi(-t)\rangle_n + 12\langle\phi(-2t)\rangle_n.$$

Now as in our calculations above, each term here is zero except $\langle\phi(0)\rangle_n = -\frac{1}{3}$, hence $\langle u_2^{(3)}(t)\rangle_n = -\frac{13}{9}$.

From equation (3.11)

$$\langle u_2^{(2)}(t)\rangle_n = -2\sum_{\ell=0}^{\infty} 4^{-\ell}\left(\langle\phi((2.4^\ell+2)t)\rangle_n + 2\langle\phi(2.4^\ell t)\rangle_n - 5\langle\phi((2.4^\ell-1)t)\rangle_n\right.$$
$$+\langle\phi(2t)\rangle_n + 2\langle\phi(0)\rangle_n - 5\langle\phi(-t)\rangle_n$$
$$\left.-2\langle\phi((-4^\ell+2)t)\rangle_n - 4\langle\phi(-4^\ell t)\rangle_n + 10\langle\phi((-4^\ell-1)t)\rangle_n\right).$$

Once more, each term here is zero except $\langle\phi(0)\rangle_n$, hence $\langle u_2^{(2)}(t)\rangle_n = \frac{4}{3}\sum_{\ell=0}^{\infty} 4^{-\ell} = \frac{16}{9}$.

For $\langle u_2^{(1)}(t)\rangle_n$, looking at (3.10) we see that once again every average will be zero except that involving $\phi(0)$. (There will never be the necessary pure differences of powers of 4.) We have

$$\langle u_2^{(1)}(t)\rangle_n = \tfrac{3}{2}\sum_{\ell,m=0}^{\infty} 4^{-(\ell+m)}\langle\phi(0)\rangle_n = -\tfrac{1}{2}\sum_{\ell,m=0}^{\infty} 4^{-(\ell+m)} = -\tfrac{8}{9}$$

Combining these three results (using equation (5.4) and the fact that $A = \gamma - 1$) we deduce that

$$\frac{\langle C\rangle_n}{A} = -\frac{5}{9}.$$

Appendix B. $\langle B\bar{B}\rangle_n$

We have

$$\langle B\bar{B}\rangle_n = \langle u_1(t+\omega)\bar{u}_1(t+\omega)\rangle_n - \gamma\langle u_1(t+\omega)\bar{u}_1(t)\rangle_n - \bar{\gamma}\langle\bar{u}_1(t+\omega)_1 u(t)\rangle_n + \langle u_1(t)\bar{u}_1(t)\rangle_n .$$

For the calculation it will suffice to consider $\langle u_1(t+\omega)\bar{u}_1(t)\rangle_n$.

$$\langle u_1(t+\omega)\bar{u}_1(t)\rangle_n = \tfrac{1}{4}\sum_{\ell,m=0}^{\infty} 4^{-(\ell+m)}\langle\left(e^{-2\pi i 4^\ell 2\omega} e(2.4^\ell) + e(0) - 2\,e^{2\pi i 4^\ell\omega} e(-4^\ell)\right)$$
$$\times\left(e(-2.4^m) + e(0) - 2e(4^m)\right)\rangle_n$$
$$= \tfrac{1}{4}\sum_{\ell,m=0}^{\infty} 4^{-(\ell+m)}\left(e^{-2\pi i 4^\ell 2\omega}\left(\langle e(2.4^\ell - 2.4^m)\rangle_n + \langle e(2.4^\ell)\rangle_n - 2\langle e(2.4^\ell + 4^m)\rangle_n\right)\right.$$
$$+\langle e(-2.4^m)\rangle_n + \langle e(0)\rangle_n - 2\langle e(4^m)\rangle_n$$
$$\left.-2\,e^{2\pi i 4^\ell\omega}\left(\langle e(-4^\ell - 2.4^m)\rangle_n + \langle e(-4^\ell)\rangle_n - 2\langle e(-4^\ell + 4^m)\rangle_n\right)\right).$$

Unlike the previous averages, we now have some non-trivial non-zero terms. Using the fact that $4^\ell - 4^m \equiv 0 \pmod{4^n - 1}$ if and only if $\ell \equiv m \pmod{n}$ we have

$$\langle u_1(t+\omega)\bar{u}_1(t)\rangle_n = \tfrac{1}{4}\sum_{\ell,m=0}^{\infty} 4^{-(\ell+m)} + \tfrac{1}{4}\sum_{\substack{\ell,m=0\\ \ell\equiv m\,(\mathrm{mod}\,n)}}^{\infty} 4^{-(\ell+m)}\left(e^{-2\pi i 4^\ell 2\omega} + 4\,e^{2\pi i 4^\ell\omega}\right).$$

Defining the function h by

$$h(x) = \sum_{m=0}^{\infty} 4^{-2m} e^{-2\pi i 4^m x}$$

we may write

$$\langle u_1(t+\omega)\bar{u}_1(t)\rangle_n = \frac{1}{4}\left(\frac{4}{3}\right)^2 + \frac{1}{4}\left(\frac{4^n+1}{4^n-1}\right)(h(2\omega) + 4h(-\omega)).$$

Using equation (5.4), and setting $\omega = 0$ when appropriate, we deduce that

$$\langle B\bar{B}\rangle_n = \tfrac{8}{9}\left(1 - \tfrac{1}{2}(\gamma + \bar{\gamma})\right) + \left(\frac{4^n+1}{4^n-1}\right)\left(\frac{8}{3} - \frac{\gamma}{4}(h(2\omega) + 4h(-\omega)) - \frac{\bar{\gamma}}{4}(h(-2\omega) + 4h(\omega))\right).$$

Now we wish to know the asymptotics of $\langle B\bar{B}\rangle_n$ for large n. Recalling that $\omega = 1/(4^n - 1) = 1/N$, we have

$$1 - \tfrac{1}{2}(\gamma + \bar{\gamma}) = 1 - \cos 2\pi\omega = O(1/N^2)$$

and

$$\left(\frac{4^n+1}{4^n-1}\right) = \frac{N+2}{N} = 1 + O(1/N).$$

Noting that $\frac{8}{3} = \frac{5}{2}\sum_{m=0}^{\infty} 4^{-2m}$, we have, after a little manipulation,

$$\frac{8}{3} - \frac{\gamma}{4}(h(2\omega) + 4h(-\omega)) - \frac{\bar{\gamma}}{4}(h(-2\omega) + 4h(\omega)) = \sum_{m=0}^{\infty} 4^{-2m}\left(\tfrac{1}{2}(1 - \cos(2\pi\omega(2.4^m - 1)) + 2(1 - \cos(2\pi\omega(4^m + 1))\right).$$

To understand the behaviour of this sum we split it into a finite part and an infinite part at some point to be specified later. Write

$$\sum_{m=0}^{\infty} = \sum_{m=0}^{M-1} + \sum_{m=M}^{\infty} = \Sigma_1 + \Sigma_2$$

say. Now

$$|\Sigma_2| \leqslant 2\sum_{m=M}^{\infty} 4^{-2m}\left(\tfrac{1}{2} + 2\right) = \tfrac{16}{3} 4^{-2M}.$$

Expanding the cosine series and gathering together terms we get

$$\Sigma_1 = 8\pi^2 M\omega^2 + O(\omega^2 4^{-M}).$$

We therefore choose M so that $\omega = 4^{-M}$, i.e. so that $M = \log N/\log 4$, and see

$$\Sigma_1 = \frac{8\pi^2 \log N}{N^2 \log 4} + O(1/N^3)$$

and

$$|\Sigma_2| = O(1/N^2)\,.$$

Hence

$$\langle B\bar{B}\rangle_n = \frac{8\pi^2 \log N}{N^2 \log 4} + O(1/N^2)$$

and so, recalling the fact that $A\bar{A} = 4\pi^2/N^2 + O(1/N^4)$,

$$\frac{\langle B\bar{B}\rangle_n}{A\bar{A}} = \frac{\log N}{\log 2} + O(1)\,.$$

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